DYNAMICAL ENTROPY OF QUANTUM RANDOM WALKS

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Abstract
In this manuscript, we study discrete-time dynamics of systems that arise in physics and information theory, and the measure of disorder in these systems known as dynamical entropy. The study of dynamics in classical systems is done from two distinct viewpoints: random walks and dynamical systems. Random walks are probabilistic in nature and are described by stochastic processes. On the other hand, dynamical systems are described algebraically and deterministic in nature. The measure of disorder from either viewpoint is known as dynamical entropy.

Entropy is an essential notion in physics and information theory. Motivated by the study of disorder for the positions and velocities of gas molecules, the notion of entropy was first introduced mathematically by Boltzmann near the end of the 19th century and gives rise to the second law of thermodynamics. Almost eighty years after Boltzmann, Shannon became the father of the new field of information theory when he produced his groundbreaking works where he used entropy as a measure of information transfer between two sources. In the last two years of the 1950’s, Kolmogorov and Sinai extended the notions of Boltzmann to a dynamical entropy. The Kolmogorov-Sinai dynamical entropy gives a measure for the disorder of a system of particles (e.g. gas molecules) averaged over time, quantifying the uncertainty in the dynamics of a system.

The advent of quantum mechanics and its pervasiveness in nature has required the development of non-commutative generalizations of dynamics and dynamical entropy to the quantum regime. Many of each have been proposed. In particular, we recall the definitions of quantum random walks, dynamical systems and Markov chains. We motivate each generalization by relating to its classical counterpart. Quantum dynamical entropy (QDE) is a generalization of the Kolmogorov-Sinai dynamical entropy to quantum mechanics. There have been numerous definitions for QDE beginning with that of Connes, Narnhofer and Thirring in 1987. We focus on the semi-classical approach given by
Słomczyński and Życzkowski in 1994 and the quantum Markov chain approach which started with Accardi, Ohya and Watanabe in 1997.

Linearity of a dynamical entropy means that the dynamical entropy of the \( n \)-fold composition of a dynamical map with itself is equal to \( n \) times the dynamical entropy of the map for every positive integer \( n \). We show that the quantum dynamical entropy introduced by Słomczyński and Życzkowski is nonlinear in the time interval between successive measurements of a quantum dynamical system. This is in contrast to Kolmogorov-Sinai dynamical entropy for classical dynamical systems, which is linear in time. We also compute the exact values of quantum dynamical entropy for the Hadamard walk with varying Lüders-von Neumann instruments and partitions.

In 1948, Shannon proved the Source Coding Theorem which gives upper and lower bounds on the minimal expected codeword length in terms of the entropy of a random variable. This theorem can be leveraged to give the minimal expected average codeword length for a string of symbols in terms of entropy rate, which can be interpreted as the dynamical entropy of a stochastic process. In 1994, Schumacher defined indeterminate-length quantum codes and proved a Quantum Source Coding Theorem. We introduce the notion of stochastic ensembles of pure states and give a novel representation in terms of quantum Markov chains. Moreover, this representation allows us to extend the Quantum Source Coding Theorem, giving the minimal expected average codeword length of an indeterminate-length quantum codes in terms of quantum dynamical entropy.

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